Cyclic pattern containment and avoidance

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A R T I C L E   I N F O

A B S T R A C T

The study of pattern containment and avoidance for linear permutations is a well-established area of enumerative combinatorics. A cyclic permutation is the set of all rotations of a linear permutation. Vella and Callan independently initiated the study of permutation avoidance in cyclic permutations and characterized the avoidance classes for all single permutations of length 4. We continue this work. In particular, we derive results about avoidance of multiple patterns of length 4, and we determine generating functions for the cyclic descent statistic on these classes. We also consider consecutive pattern containment, and relate the generating functions for the number of occurrences of certain linear and cyclic patterns. Finally, we end with various open questions and avenues for future research.

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1. Introduction

We first review some notions from the well-studied theory of patterns in (linear) permutations. More information on this topic can be found in the texts of Bóna [8], Sagan [36], or Stanley [38,39]. Let $\mathbb{N}$ be the nonnegative integers. If $m, n \in \mathbb{N}$ then we define $[m, n] = \{m, m + 1, \ldots, n\}$ which we abbreviate to $[n] = [1, n]$ when $m = 1$. Consider the symmetric group $\mathfrak{S}_n$ of all permutations $\pi = \pi_1 \pi_2 \ldots \pi_n$ of $[n]$ written in one-line notation. We call n the *length* of π and write $|\pi| = n$. We will sometimes put commas between the elements of π for readability. We say that two sequences of distinct integers $\pi = \pi_1 \ldots \pi_k$ and $\sigma = \sigma_1 \ldots \sigma_k$ are *order isomorphic*, written $\pi \cong \sigma$, whenever $\pi_i < \pi_j$ if and only if $\sigma_i < \sigma_j$. If $\sigma \in \mathfrak{S}_n$ and $\pi \in \mathfrak{S}_k$ then σ *contains* π as a *pattern* if there is a subsequence σ' of σ with $|\sigma'| = k$ and $\sigma' \cong \pi$. If no such subsequence exists then σ *avoids* π. We use the notation

$$\mathrm{Av}_n(\pi) = \{\sigma \in \mathfrak{S}_n \mid \sigma \text{ avoids } \pi\}$$

for the *avoidance class* of π. For example $\sigma = 42351$ contains the pattern $\pi = 3241$ because of the subsequence 4251 among others. But it avoids 1234 because it has no increasing subsequence of length 4. One can extend this notion to sets of permutations Π by letting

$$\mathrm{Av}_n(\Pi) = \{\sigma \in \mathfrak{S}_n \mid \sigma \text{ avoids all } \pi \in \Pi\} = \bigcap_{\pi \in \Pi} \mathrm{Av}_n(\pi).$$

A famous theorem of Erdős and Szekeres [26] can be stated in terms of pattern containment and avoidance. Let

$$\iota_n = 12 \ldots n$$

and

$$\delta_n = n \ldots 21$$

be the increasing and decreasing permutations of length n, respectively.

Theorem 1.1 *([26]). Suppose $m, n \in \mathbb{N}$. Then any $\sigma \in \mathfrak{S}_{mn+1}$ contains either ι_{m+1} or δ_{n+1}. This is the best possible in that there exist permutations in $\mathfrak{S}_{mn}$ which avoid both ι_{m+1} and δ_{n+1}.* $\square$

The *diagram* of $\pi \in \mathfrak{S}_n$ is the collection of points (i, π_i) in the first quadrant of the Cartesian plane. The graphical representation of $\pi = 42351$ is given on the left in Fig. 1. It follows that we can act on π with the dihedral group of the square

$$D_4 = \{\rho_0, \rho_{90}, \rho_{180}, \rho_{270}, r_0, r_1, r_{-1}, r_\infty\}$$
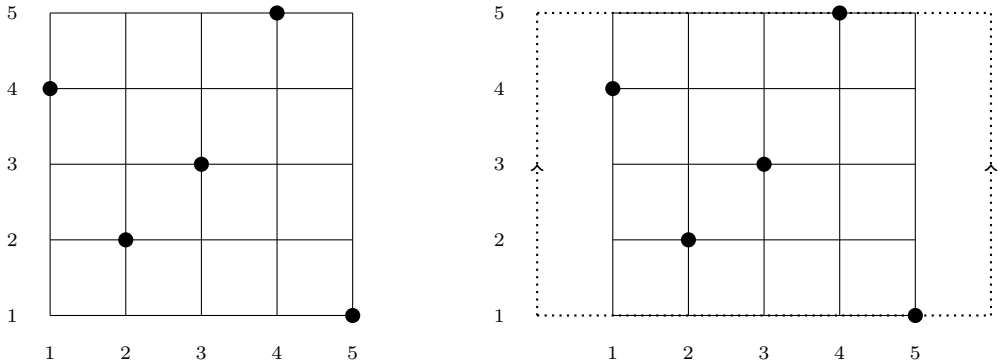
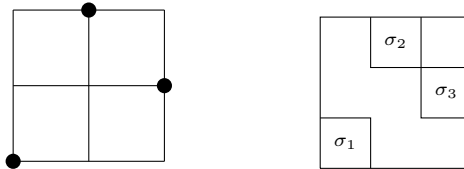**Fig. 1.** The graph of 42351 on the left and of [42351] on the right.

Fig. 2. The diagram of 132 (left) and $132\langle\sigma_1, \sigma_2, \sigma_3\rangle$ (right).

where ρ_θ is rotation counterclockwise through θ degrees and r_m is reflection in a line of slope m. We wish to write some of these rigid motions in terms of the one-line notation for $\pi = \pi_1\pi_2\ldots\pi_n$. Reflection in a vertical line gives the *reversal* of π which is

$$\pi^r = \pi_n \ldots \pi_2\pi_1.$$

Similarly, reflection in a horizontal line results in the *complement* of π

$$\pi^c = n+1 - \pi_1, \quad n+1 - \pi_2, \quad \ldots, \quad n+1 - \pi_n.$$

Combining these two operations gives rotation by 180 degree or *reverse complement*

$$\pi^{rc} = n+1 - \pi_n, \quad \ldots, \quad n+1 - \pi_2, \quad n+1 - \pi_1.$$

We apply any of these operations to sets of permutations by applying them to each element of the set.

We can use diagrams to inflate permutations. If we are given $\pi = \pi_1\pi_2\ldots\pi_n \in \mathfrak{S}_n$ and permutations $\sigma_1, \sigma_2, \ldots, \sigma_n$ then the *inflation of π by the σ_i* is the permutation $\pi\langle\sigma_1, \sigma_2, \ldots, \sigma_n\rangle$ whose diagram is obtained from that of π by replacing each vertex (i, π_i) by a copy of σ_i. For example, given $\pi = 132$ and $\sigma_1, \sigma_2, \sigma_3$ then a schematic of the diagram of $132\langle\sigma_1, \sigma_2, \sigma_3\rangle$ is given on the right in Fig. 2. More concretely, if $\sigma_1 = 21$, $\sigma_2 = 1$, and $\sigma_3 = 213$ then

$$132\langle\sigma_1, \sigma_2, \sigma_3\rangle = 216435.$$

We say that patterns π and π' are *Wilf equivalent*, written $\pi \equiv \pi'$, if $\# \operatorname{Av}_n(\pi) = \# \operatorname{Av}_n(\pi')$ for all $n \in \mathbb{N}$ where the hash symbol denotes cardinality. This definition extends in the obvious way to sets of patterns. Note that if π and π' are Wilf equivalent then both must be in the same $\mathfrak{S}_n$. It is easy to see that if $\phi \in D_n$ then $\pi \equiv \phi(\pi)$ and so these are called *trivial Wilf equivalences*. As is well known, all elements of $\mathfrak{S}_3$ are Wilf equivalent.

Theorem 1.2. *If $\pi \in \mathfrak{S}_3$ then*

$$\# \operatorname{Av}_n(\pi) = C_n$$

where $C_n = \frac{1}{n+1}\binom{2n}{n}$ it the nth Catalan number. $\quad\square$

Trivial Wilf equivalence carries over to sets Π of permutations. Simion and Schmidt [37] determined all Wilf equivalences among the $\operatorname{Av}_n(\Pi)$ for all $\Pi \subseteq \mathfrak{S}_3$.

A *permutation statistic* is a map $\operatorname{st} : \uplus_{n \geq 0} \mathfrak{S}_n \to S$ where S is some set. Many statistics are based on the *descent set* statistic which is

$$\operatorname{Des} \pi = \{i \mid \pi_i > \pi_{i+1}\}.$$

The elements $i \in \operatorname{Des} \pi$ are called *descents* and if $\pi_i < \pi_{i+1}$ then i is called an *ascent*. Four famous statistics related to Des are the *descent number* statistic

$$\operatorname{des} \pi = \# \operatorname{Des} \pi$$

the *major index* statistic

$$\operatorname{maj} \pi = \sum_{i \in \operatorname{Des} \pi} i,$$

the *inversion* statistic

$$\operatorname{inv} \pi = \#\{(i, j) \mid i < j \text{ and } \pi_i > \pi_j\},$$

and the *exceedance statistic*

$$\operatorname{exc} \pi = \#\{i \mid \pi(i) > i\}.$$

Let st be a statistic whose range is $\mathbb{N}$ and q be a variable. If Π is a set of patterns then its avoidance class has a corresponding generating function

$$F_n^{\operatorname{st}}(\Pi) = F_n^{\operatorname{st}}(\Pi; q) = \sum_{\sigma \in \operatorname{Av}_n(\Pi)} q^{\operatorname{st} \sigma}.$$

Say that Π and Π' are st-*Wilf equivalent* and write $\Pi \overset{\mathrm{st}}{\equiv} \Pi'$ if $F_n^{\mathrm{st}}(\Pi) = F_n^{\mathrm{st}}(\Pi')$ for all $n \geq 0$. Clearly st-Wilf equivalence implies Wilf equivalence. The maj- and inv-Wilf equivalence classes for $\Pi \subseteq \mathfrak{S}_3$ were determined by Dokos, Dwyer, Johnson, Sagan, and Selsor [17].

If $\pi = \pi_1 \pi_2 \ldots \pi_n \in \mathfrak{S}_n$ then the corresponding *cyclic permutation* is the set of all rotations of π, denoted by

$$[\pi] = \{\pi_1 \pi_2 \ldots \pi_n, \quad \pi_2 \ldots \pi_n \pi_1, \quad \ldots, \quad \pi_n \pi_1 \ldots, \pi_{n-1}\}.$$

These are sometimes called *horizontal rotations* in the literature to distinguish them from vertical rotations of the diagram [24,25]. Our notion of cyclic permutations has appeared in the literature under different names: Callan [10] calls them "circular permutations", and Vella [41] calls them "cyclic arrangements". We also note that some authors use the term "cyclic permutation" to refer instead to a linear permutation whose disjoint cycle decomposition is a single cycle, and study pattern avoidance in this setting [2,21]. Continuing our example from the beginning of the section,

$$[42351] = \{42351, \quad 23514, \quad 35142, \quad 51423, \quad 14235\}.$$

If necessary, we will call permutations from $\mathfrak{S}_n$ *linear* to distinguish them from their cyclic cousins. We also use square brackets to denote cyclic analogues of objects defined in the linear case. For example, $[\mathfrak{S}_n]$ is the set of all cyclic permutations of length n. We say a cyclic permutation $[\sigma]$ *contains* $[\pi]$ *as a pattern* if there is some rotation σ' of σ which contains π linearly. Otherwise $[\sigma]$ *avoids* $[\pi]$. In our perennial example, even though 42351 avoids 1234 we have that $[42351]$ contains $[1234]$ since the rotation 14235 has the copy 1235 of this pattern. Given a set $[\Pi]$ of cyclic patterns, the cyclic avoidance class $\mathrm{Av}_n[\Pi]$ is defined as expected. Note that when using a specific set of cyclic permutations, the square brackets will be put around the permutations themselves, for example, $\mathrm{Av}_n([\pi], [\pi'])$.

One can also put certain restrictions on the form of a copy of a pattern. A *vincular pattern* $[\pi]$ is one where certain cyclically adjacent elements of π are required to be cyclically adjacent in any copy. In this case, the adjacent elements are underlined. For example, $[42351]$ contains the vincular pattern $[1\underline{32}4]$ because the copy $[1\underline{42}5]$ has the three corresponding elements cyclically adjacent. However, it avoids $[\underline{132}4]$ because neither of the two copies of $[1324]$ have the prescribed adjacencies. Call a vincular pattern of the form $[\underline{\pi_1 \ldots \pi_k}]$ *consecutive*. See Elizalde's survey article [20] for further information about consecutive pattern containment and avoidance in the linear case.

Vella [41] is the first person, to our knowledge, to consider (nonvincular) cyclic pattern avoidance and calculate $\#\mathrm{Av}_n[1243]$ and $\#\mathrm{Av}_n[1324]$. Callan [10] determined $\#\mathrm{Av}_n[\pi]$ for all $[\pi] \in [\mathfrak{S}_4]$. Gray, Lanning, and Wang continued work in this direction considering cyclic packing of patterns [27] and patterns in colored cyclic permutations [28]. The study of vincular patterns in the linear case was originated by Babson and Steingrímsson [3]. More recently, and inspired by the present work, Li [31] studied avoiding sets of

vincular patterns of length three and four. One of the cases left open by Li was resolved by Mansour and Shattuck [33]. Menon and Singh [35] have also built on our work by considering avoidance of a pair of patterns, one of length 4 and the other of length $k \geq 4$.

A cyclic version of the Erdős–Szekeres Theorem was proved by Czabarka and Wang [13] and will be useful for us in the sequel.

Theorem 1.3 *([13]). If $m, n \in \mathbb{N}$ then any $[\sigma] \in [\mathfrak{S}_{mn+2}]$ contains either $[\iota_{m+2}]$ or $[\delta_{n+2}]$. This is the best possible in that there exist permutations in $[\mathfrak{S}_{mn+1}]$ which avoid both $[\iota_{m+2}]$ and $[\delta_{n+2}]$.* □

The graph of a cyclic permutation $[\pi]$ is obtained by embedding the graph of π on a cylinder. This is indicated on the right in Fig. 1 by identifying the two dotted arrows. Cyclic Wilf equivalence has the obvious definition. But note that now there are fewer trivial cyclic Wilf equivalences since we need the chosen group element to preserve the cylinder, not just the square. So the only trivial equivalences are

$$[\pi] \equiv [\pi^r] \equiv [\pi^c] \equiv [\pi^{rc}]. \tag{1}$$

Certain linear permutation statistics have obvious cyclic analogues. For example, if $\pi \in \mathfrak{S}_n$ then its *cyclic descent number* is

$$\text{cdes}[\pi] = \#\{i \mid \pi_i > \pi_{i+1} \text{ where subscripts are taken modulo } n\}.$$

Note that this is well defined because the cardinality does not depend on which representative of $[\pi]$ is chosen. To illustrate, $\pi = 23514$ has cyclic descents at indices 3 and 5 so $\text{cdes}[\pi] = 2$. The corresponding generating function $F_n^{\text{cdes}}[\Pi]$ where $[\Pi]$ is a set of cyclic permutations, and cdes-Wilf equivalence should now need no definition. Note that cdes is another form of the exceedance statistic on linear permutations. In particular, if $\pi = \pi_1 \pi_2 \ldots \pi_n$ then

$$\text{cdes}[\pi] = \text{exc}(\pi_n, \ldots, \pi_2, \pi_1)$$

where $(\pi_n, \pi_{n-1} \ldots, \pi_1)$ is cycle notation for the linear permutation which, as a function, sends π_i to π_{i-1} for all i modulo n.

The rest of this paper is organized as follows. Section 2 will extend Callan's work by enumerating $\text{Av}_n[\Pi]$ for $[\Pi] \subset [\mathfrak{S}_4]$ consisting of two or more patterns. One of our principle proof techniques will be the use of generating trees. In Section 3 we will compute the cyclic descent generating functions for $[\Pi] \subset [\mathfrak{S}_4]$, thus refining the previous enumerations. Section 4 will be devoted to the study of consecutive patterns whose initial element is 1. We will show that there is a simple relationship between the generating functions counting the number of occurrences of a consecutive pattern in linear permutations and its cyclic analogue. This will be used to resolve two conjectures in an earlier version of this article which were also proved in the aforementioned paper of Li [31]. We will end with a section of open problems and additional comments.

Table 1

Wilf equivalence classes and cardinalities of $\mathrm{Av}_n[\Pi]$ for certain $[\Pi]$ and $n \geq 5$.

$[\Pi]$	$\# \mathrm{Av}_n[\Pi]$
$\{[1234]\}, \quad \{[1432]\}$	$2^n + 1 - 2n - \binom{n}{3}$
$\{[1243]\}, \quad \{[1342]\}$	$2^{n-1} - n + 1$
$\{[1324]\}, \quad \{[1423]\}$	F_{2n-3}
$\{[1234],[1243]\}, \quad \{[1234],[1324]\}, \quad \{[1234],[1342]\}, \quad \{[1243],[1432]\}$ $\{[1342],[1432]\}, \quad \{[1423],[1432]\}$	$2(n-2)$
$\{[1234],[1423]\}, \quad \{[1243],[1324]\}, \quad \{[1243],[1423]\}, \quad \{[1324],[1342]\}$ $\{[1324],[1432]\}, \quad \{[1342],[1423]\}$	$1 + \binom{n-1}{2}$
$\{[1234],[1432]\}$	0
$\{[1243],[1342]\}$	4
$\{[1324],[1423]\}$	2^{n-2}
$\{[1234],[1243],[1324]\}, \quad \{[1234],[1324],[1342]\}, \quad \{[1243],[1324],[1342]\}$ $\{[1243],[1342],[1423]\}, \quad \{[1243],[1423],[1432]\}, \quad \{[1342],[1423],[1432]\}$	3
$\{[1234],[1243],[1342]\}, \quad \{[1243],[1342],[1432]\}$	2
$\{[1234],[1243],[1423]\}, \quad \{[1234],[1324],[1423]\}, \quad \{[1234],[1342],[1423]\}$ $\{[1243],[1324],[1423]\}, \quad \{[1243],[1324],[1432]\}, \quad \{[1324],[1342],[1423]\}$ $\{[1324],[1342],[1432]\}, \quad \{[1324],[1423],[1432]\}$	$n-1$
$\{[1234],[1243],[1432]\}, \quad \{[1234],[1324],[1432]\}$ $\{[1234],[1342],[1432]\}, \quad \{[1234],[1423],[1432]\}$	0
$\{[1234],[1243],[1324],[1342]\}, \quad \{[1243],[1342],[1423],[1432]\}$	1
$\{[1234],[1243],[1324],[1423]\}, \quad \{[1234],[1243],[1342],[1423]\}$ $\{[1234],[1324],[1342],[1423]\}, \quad \{[1243],[1324],[1342],[1423]\}$ $\{[1243],[1324],[1342],[1432]\}, \quad \{[1243],[1324],[1423],[1432]\}$ $\{[1324],[1342],[1423],[1432]\}$	2
$\{[1234],[1243],[1324],[1342],[1423]\}, \quad \{[1243],[1324],[1342],[1423],[1432]\}$	1

2. Pattern avoidance for nonsingleton sets

In this section we will enumerate $\mathrm{Av}_n[\Pi]$ for all $[\Pi] \subset [\mathfrak{S}_4]$ with $\#[\Pi] \geq 2$. Any cyclic Wilf equivalences stated without proof are trivial. We will collect our results, as well as those of Callan, in Table 1.

Let us first mention the simplest singleton avoidance classes where $[\pi] \in [\mathfrak{S}_k]$ for $k < 4$. In $[\mathfrak{S}_2]$ there is only one cyclic permutation $[12]$ and it is easy to see that every $[\sigma]$ of length at least 2 contains it. In $[\mathfrak{S}_3]$ there are only the patterns $[123]$ and $[321]$, and these are only avoided by $[\delta_n]$ and $[\iota_n]$, respectively.

Callan [10] enumerated $\mathrm{Av}_n[\pi]$ for any given $[\pi] \in [\mathfrak{S}_4]$. Recall the version of the Fibonacci numbers defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. Unlike the case of linear permutations in $\mathfrak{S}_3$, there are no nontrivial Wilf equivalences.

Theorem 2.1 ([10]). *For $n \geq 2$ we have*

$$\# \mathrm{Av}_n[1234] = \# \mathrm{Av}_n[1432] = 2^n + 1 - 2n - \binom{n}{3},$$

$$\# \mathrm{Av}_n[1243] = \# \mathrm{Av}_n[1342] = 2^{n-1} - n + 1,$$

$$\# \mathrm{Av}_n[1324] = \# \mathrm{Av}_n[1423] = F_{2n-3}. \quad \square$$

In presenting the enumerations for doubletons, we make the following conventions to facilitate locating a given result. All cyclic patterns will be listed starting with 1. And all sets of cyclic patterns will be given in lexicographic order. We will also use terms like "just before" or "just after" in $[\sigma]$ to refer the left-to-right order on the cylinder of a cyclic permutation in the form of Fig. 1. For example, in $[\sigma] = [42351]$ the 5 comes just before 1 and the 4 just after. We also say that an element x is between y and z if it is in the subsequence of $[\sigma]$ traversed going left-to-right around the cylinder from y to z. Continuing our example, between 2 and 5 we have 3, while between 5 and 2 we have 1 and 4.

One of our tools will be generating trees. To the best of our knowledge, these trees were introduced by Chung, Grahamm, Hoggatt, and Kleiman [12] for studying Baxter permutations. Since then, they have become an integral technique in the theory of pattern avoidance [4,9,30,42,43]. The *generating tree* for an avoidance class $\mathrm{Av}[\Pi]$, denoted by $T[\Pi]$, has as its root the permutation [12]. The children of any $[\sigma] \in \mathrm{Av}_n[\Pi]$ are all the $[\sigma'] \in \mathrm{Av}_{n+1}[\Pi]$ which can be formed by inserting $n + 1$ into one of the spaces of $[\sigma]$. A space, also called a *site*, where insertion of $n+1$ produces a permutation of the avoidance class is called *active* while the other spaces are *inactive*. A useful observation is that if a space is inactive it must be because inserting $n + 1$ there results in copy of a forbidden pattern $[\pi]$ where $n + 1$ plays the role of the largest element of π. Once we have picked a representative $\sigma = \sigma_1\sigma_2\ldots\sigma_n$ for $[\sigma]$ we will label the spaces as $1, 2, \ldots, n$ left to right where space i comes between σ_i and σ_{i+1}. The nodes for $\mathrm{Av}_n[\Pi]$ will be said to be at *level n* in $T[\Pi]$. We call the number of children of a vertex its *degree* which is denoted $\deg[\sigma]$. Given $d \in \mathbb{N}$, suppose that every cyclic permutation with $\deg[\sigma] = d$ has children of degrees $c_1, c_2, \ldots, c_d$. Then this is denoted by the *production rule*

$$(d) \to (c_1)(c_2)\ldots(c_d).$$

There may be other nodes having some special characteristic X which always produces nodes having characteristics $Y_1, Y_2, \ldots, Y_d$ which correspond to a production rule

$$(X) \to (Y_1)(Y_2)\ldots(Y_d).$$

In particular, the characteristic of being the root of the tree is denoted in a production rule by $(*)$. We can also have production rules which mix numbers for degrees and letters for characteristics. If $T[\pi]$ can be characterized by production rules, these can often be used to calculate $\# \mathrm{Av}_n[\Pi]$.

Theorem 2.2. *We have*

$$\{[1234], [1243]\} \equiv \{[1234], [1342]\} \equiv \{[1243], [1432]\} \equiv \{[1342], [1432]\}.$$

And for $n \geq 3$

$$\# \mathrm{Av}_n([1234], [1342]) = 2(n - 2).$$

Proof. We claim that $T = T([1234], [1342])$ has the following production rules

$$(*) \rightarrow (2)(2),$$
$$(1) \rightarrow (1),$$
$$(2) \rightarrow (1)(2).$$

Once these are proved then the enumeration follows easily since one can inductively show that, for $n \geq 3$, level n consists of two nodes of degree 2 and $2(n-3)$ nodes of degree 1.

It is easy to check the production rule at levels $n = 2$ and 3, so we assume that $n \geq 4$ and also that $[\sigma] \in \mathrm{Av}_n([1234], [1342])$. First of all, note that the site before n is always active. For if it were not then the result $[\sigma']$ of inserting $n + 1$ would have a copy κ of one of the patterns containing $n + 1$. But n can not be in κ since neither of the patterns have 4 followed immediately in the cycle by 3. So replacing $n + 1$ by n in κ would give a forbidden pattern in $[\sigma]$ which is a contradiction. Thus every $[\sigma]$ at has at least one child. Also σ has at most two children. For suppose

$$\sigma' = n + 1, \rho, n, \tau$$

is the result of inserting $n + 1$ in σ. It follows that $|\rho| \leq 1$ since if $\rho \geq 2$ then $[\sigma']$ has a copy of either $[4123]$ or $[4213]$. Thus $n + 1$ must be inserted either directly before n or two elements before n.

Now consider

$$\delta = n, n - 1, \ldots, 3, 2, 1, \quad \text{and} \quad \epsilon = n, n - 1, \ldots, 3, 1, 2. \tag{2}$$

It is easy to check that both sites n and $n - 1$ are active in these permutations and so both have degree 2. It is also obvious that if one inserts $n + 1$ in site n in either permutation then one gets another permutation of the same form.

From what we have done, we can finish the proof if we show that $\deg[\sigma] = 2$ implies $[\sigma] = [\delta]$ or $[\sigma] = [\epsilon]$. Write

$$\sigma = n\rho m$$

where m is the last element of σ and ρ is everything between n and m. Since $\deg[\sigma] = 2$, site $n - 1$ is active and inserting $n + 1$ there yields

$$\sigma' = n, \rho, n + 1, m.$$

Then $m \leq 2$ since otherwise $[\sigma]$ contains a copy of $[4123]$ or $[4213]$ because $n \geq 4$. In the case $m = 1$ we must have ρ decreasing. For if there is an ascent $x < y$ in ρ then $[\sigma']$ contains $[x, y, n + 1, 1]$ which is a copy of $[2341]$, a contradiction. So in this case $\sigma = \delta$. The other possibility is that $m = 2$. This forces the last element of ρ to be 1.

For if 1 is elsewhere and x is the last element of ρ then $[\sigma']$ contains $[1, x, n + 1, 2]$ which is contradictory copy of $[1342]$. Similarly to the first case, one can now show that ρ is decreasing and so $\sigma = \epsilon$ as desired. $\square$

Comparing our next result with the previous one will provide our first nontrivial Wilf equivalence.

Theorem 2.3. *We have*

$$\{[1234], [1324]\} \equiv \{[1423], [1432]\}.$$

And for $n \geq 3$

$$\# \operatorname{Av}_n([1234], [1324]) = 2(n - 2).$$

Proof. Let D stand for the decreasing permutation and E for the decreasing permutation with its largest two elements swapped. We consider the root $[12]$ to be of type D. We will show that $T = T([1234], [1324])$ has production rules

$$(1) \rightarrow (1),$$
$$(D) \rightarrow (D)(E),$$
$$(E) \rightarrow (1)(1).$$

It follows by induction that level $n \geq 3$ of T has a D, an E, and $2(n - 3)$ nodes of degree one, proving the theorem.

The same demonstration as in the previous theorem shows that the site before n in any $[\sigma] \in \operatorname{Av}_n([1234], [1324])$ is active. So again, every such permutation has at least one child. Also, every $[\sigma]$ has at most two children. Indeed, write

$$\sigma = 1\sigma_2 \ldots \sigma_n \tag{3}$$

and put $n + 1$ in site $i \geq 3$. Then $1, \sigma_2, \sigma_3, n + 1$ is a copy of either 1234 or 1324, a contradiction.

Now consider permutations corresponding to D and E at level n

$$\delta = 1, n, n - 1, n - 2, n - 3, \ldots, 2 \quad \text{and} \quad \epsilon = 1, n - 1, n, n - 2, n - 3, \ldots, 2. \tag{4}$$

It is easy to check that both sites 1 and 2 are active in δ, ϵ. So, by the previous paragraph, they both have degree 2. Furthermore, the two children of δ have the form D and E.

We will be done if we can show that $[\sigma]$ having two children implies $[\sigma] = [\delta]$ or $[\epsilon]$. Write σ as in (3). Since the active sites must be 1 and 2, and the site before n must be active, either $\sigma_2 = n$ or $\sigma_3 = n$. If $\sigma_2 = n$ and there is an ascent $x < y$ in the rest of the

permutation, then after inserting $n + 1$ in position 2 we have $[x, y, n, n + 1]$ which is a copy of [1234], a contradiction. So in this case $[\sigma] = [\delta]$. Alternatively, suppose $\sigma_3 = n$. This forces $\sigma_2 = n - 1$, since if $\sigma_2 = x < n - 1$ then $n - 1$ comes after n. But inserting $n + 1$ in position 1 gives $[x, n, n - 1, n + 1]$ which is a copy of [1324]. And similarly to the first case we see that the rest of σ is decreasing. The result is that $[\sigma] = [\epsilon]$. This completes the proof. $\square$

Theorem 2.4. *We have*

$$\{[1234], [1423]\} \equiv \{[1324], [1432]\}.$$

And for $n \geq 1$

$$\# \operatorname{Av}_n([1234], [1423]) = 1 + \binom{n-1}{2}.$$

Proof. Suppose $[\sigma] \in \operatorname{Av}_n([1234], [1423])$ and write

$$\sigma = 1\rho n\tau \tag{5}$$

where ρ and τ are the subsequences between 1 and n, and between n and 1, respectively. Now ρ and τ must be decreasing since $[\sigma]$ avoids [1234] and [1423], respectively. Furthermore, ρ must consist of consecutive integers since, if not, then we have $x < y < z$ such that $1zxny$ is a subsequence of σ. So $[xnyz]$ is a copy of [1423] in $[\sigma]$, which is a contradiction. Conversely, it is easy to check that if σ has the form (5) with ρ and τ decreasing and ρ consecutive then $[\sigma] \in \operatorname{Av}_n([1234], [1423])$. So we have characterized the elements of this class.

To finish the enumeration, if $\rho = \emptyset$ there is one corresponding σ. But if $\rho \neq \emptyset$ then choosing the smallest and largest element of ρ from the elements $2, 3, \ldots, n-1$ completely determines σ. Since these two elements could be equal, we are choosing 2 elements from $n - 2$ elements with repetition which is counted by $\binom{n-1}{2}$. $\square$

The following result follows immediately from Theorem 1.3

Theorem 2.5. *We have*

$$\# \operatorname{Av}_n([1234], [1432]) = 0$$

for $n \geq 6$. $\square$

We now have, by comparison with Theorem 2.4, another nontrivial Wilf equivalence.

Theorem 2.6. *We have*

$$\{[1243], [1324]\} \equiv \{[1243], [1423]\} \equiv \{[1324], [1342]\} \equiv \{[1342], [1423]\}.$$

And for $n \geq 1$

$$\# \operatorname{Av}_n([1324], [1342]) = 1 + \binom{n-1}{2}.$$

Proof. Take $[\sigma] \in \operatorname{Av}_n([1324], [1342])$ and write σ as in (5). Then ρ is increasing since $[\sigma]$ avoids [1324]. And every element of ρ is smaller than every element of τ since $[\sigma]$ avoids [1342]. To avoid a copy of one of the forbidden patterns containing the 1 of σ we must have that τ avoids 213 and 231. And to avoid a copy of [1324] where n plays the role of 4, it must be that τ avoids 132. The τ which avoid these three patterns are exactly those which are inflations of the form $\tau = 21\langle \delta_k, \iota_l \rangle$ for some $k, l \geq 0$ (see the chart on page 2773 of [17]). Absorbing the 1 and n of σ into ρ and τ, respectively, we see that

$$\sigma = 132\langle \iota_j, \delta_k, \iota_l \rangle \tag{6}$$

where $j, k \geq 1$ and $l \geq 0$. Again, it is not hard to check that for every σ of this form we have $[\sigma] \in \operatorname{Av}_n([1324], [1342])$.

To enumerate these σ, we distinguish two cases. If $l \geq 2$ then picking the smallest and largest elements of the copy of ι_l from $2, 3, \ldots, n-1$ completely determines σ. So in this case there are $\binom{n-2}{2}$ choices. If $l \leq 1$ then the copy of ι_l can be appended to the copy of δ_k so that $\sigma = 12\langle \iota_j, \delta_{n-j} \rangle$. Since we must have 1 and n in the ascending and decreasing subsequences, there are now $n - 1$ choices. Adding the two counts given the desired result. $\square$

Theorem 2.7. *For $n \geq 4$ we have*

$$\# \operatorname{Av}_n([1243], [1342]) = 4.$$

Proof. Take $[\sigma] \in \operatorname{Av}_n([1243], [1342])$ and write σ as in (5). Then ρ and τ can not both be nonempty. For if $x \in \rho$ and $y \in \tau$ then $1xny$ is a copy of either 1243 or 1342.

Assume first that $\rho = \emptyset$ so that

$$\sigma = 1n\tau. \tag{7}$$

Then τ must be increasing or decreasing. For suppose it was neither. Then it would contain a copy of one of the patterns 132, 231, 213, or 312. In the first two cases this would give, together with the 1, a copy of 1243 or 1342 in σ. And in the last two cases, prepending n gives a copy of 4213 or 4312. Conversely, if σ is given by (7) with τ increasing or decreasing then it is easy to verify that $[\sigma] \in \operatorname{Av}_n([1243], [1342])$.

Using the same ideas, one can also show that if $\tau = \emptyset$ then one gets exactly two elements of $\operatorname{Av}_n([1243], [1342])$, of the form $\sigma = 1\rho n$ where ρ is either increasing or decreasing. Thus there are a total of four elements in the avoidance class. $\square$

Theorem 2.8. *For $n \geq 3$ we have*

$$\# \operatorname{Av}_n([1324], [1423]) = 2^{n-2}.$$

Proof. Take $[\sigma] \in \operatorname{Av}_n([1324], [1423])$ and write

$$\sigma = n\rho.$$

Now $n - 1$ either begins or ends ρ since otherwise 4132 or 4231 is a pattern in σ. One shows similarly that $n - 2$ either begins or ends ρ with $n - 1$ removed. Continuing in this manner, we see that there are 2 choices for the positions of $n - 1, n - 2, \ldots, 2$. Checking, as usual, that all such permutations are actually in the avoidance set, the enumeration follows. $\square$

For subsets of patterns $\Pi \subseteq \mathfrak{S}_4$ with three or more permutations, the structure of the avoidance classes and corresponding enumeration can easily be derived by combining the appropriate results for avoiding 2-element subsets of Π. So we omit the details and merely summarize the results in Table 1.

3. Cyclic descent generating functions

We will now consider the generating function for the number of cyclic descents over various avoidance classes $[\Pi] \subset [\mathfrak{S}_4]$, starting with those defined by a single element. We will sometimes use the characterizations given by Callan [10] for these classes to facilitate our work, and use the abbreviation

$$D_n([\Pi]) = D_n([\Pi]; q) = \sum_{\sigma \in \operatorname{Av}_n[\Pi]} q^{\operatorname{cdes} \sigma}$$

for the generating function.

To begin, we have a lemma showing that trivial Wilf equivalences also give simple relationships between the corresponding generating functions.

Lemma 3.1. *For any $[\Pi]$ we have*

$$D_n([\Pi]^r; q) = D_n([\Pi]^c; q) = q^n D_n([\pi]; 1/q)$$

and

$$D_n([\Pi]^{rc}; q) = D_n([\Pi]; q).$$

Proof. Reversing or complementing a permutation turns all cyclic descents into cyclic ascents and vice-versa. Translating this into generating functions gives the first displayed equalities. And the second displayed equation follows from the previous display. $\square$

Now consider the possible $D_n([\pi])$ for $[\pi] \in [\mathfrak{S}_4]$. We begin with the simplest case.

Theorem 3.2. *We have* $D_n([1423]; q) = q^n D_n([1324]; 1/q)$ *where, for* $n \geq 2$,

$$D_n([1324]; q) = \sum_{k=1}^{n-1} \binom{n+k-3}{n-k-1} q^k.$$

Proof. We use Callan's characterization of this avoidance class to obtain a recursion for $D_n([1324])$. If $[\sigma] \in \mathrm{Av}_n([1324])$ and $n \geq 3$ then write $\sigma = \sigma_1 \sigma_2 \ldots \sigma_{n-1} n$. Let k be the index such that $\sigma_k = n - 1$. There are two cases.

If $k = n-1$ then $\sigma = \tau, n-1, n$ where $[\tau, n-1] \in \mathrm{Av}_{n-1}([1324])$ and this is a bijection. Since $\mathrm{cdes}[\sigma] = \mathrm{cdes}[\tau, n-1]$, this case contributes $D_{n-1}([1324])$ to the recursion.

If $1 \leq k \leq n-2$ then this forces

$$\sigma = 2314[\iota_{k-1}, 1, \tau, 1]$$

for some τ such that $[\tau n]$ avoids $[1324]$. Because of the extra descent caused by $n-1$ we have $\mathrm{cdes}[\sigma] = 1 + \mathrm{cdes}[\tau n]$. So this case gives a contribution of $\sum_{k=1}^{n-2} q D_{n-k}([1324])$.

Putting everything together, we have

$$D_n([1324]) = D_{n-1}([1324]) + \sum_{k=1}^{n-2} q D_{n-k}([1324]),$$

for $n \geq 3$ and $D_2([1324]) = q$. It is now a simple manner of manipulating binomial coefficients to show that the formula given in the theorem satisfies this initial value problem. $\square$

For the next case, we will use a characterization of the class different from the one found by Callan. This will permit us to avoid the use of a recurrence.

Lemma 3.3. *Suppose* $[\sigma] \in [\mathfrak{S}_n]$ *and write* $\sigma = 1\rho n\tau$. *We have* $[\sigma] \in \mathrm{Av}_n([1342])$ *if and only if the following three conditions are satisfied:*

(a) ρ *and* τ *both avoid* $\{213, 231\}$,
(b) $\max \rho < \min \tau$,
(c) *there is not both a descent in* ρ *and an ascent in* τ.

Proof. For the forward direction, suppose $[\sigma] \in \mathrm{Av}_n([1342])$. Condition (a) is true since if either ρ or τ contains 213 then, together with n, we have that $[\sigma]$ contains $[2134]$. Similarly, if either contains 231 then $[\sigma]$ contains the forbidden pattern by prepending the 1. As far as (b), if there is $y > x$ with $y \in \rho$ and $x \in \tau$ then $[1ynx]$ is a copy of $[1342]$. Finally for (c), if there were a descent in ρ and an ascent in τ then, because of (b), putting them together would again give a copy of the pattern to avoid.

The converse is similar where one assumes that a copy of [1342] exists and then considers all the different intersections it could have with 1, ρ, n, and τ. We leave the details to the reader. $\square$

In order to use this lemma, we will need a result about the ordinary descent statistic on linear permutations avoiding $\{123, 231\}$. The next result is a specialization of Proposition 5.2 of the paper of Dokos, Dwyer, Johnson, Sagan, and Selsor [17] and so the proof is omitted.

Lemma 3.4 *([17]). We have*

$$\sum_{\sigma \in \mathrm{Av}_n(213,231)} q^{\mathrm{des}\,\sigma} = (1+q)^{n-1}. \quad \square$$

We need one last well-known definition. Call a polynomial $f(q) = \sum_{k=0}^{n} a_k q^k$ of degree n *symmetric* if $a_k = a_{n-k}$ for all $0 \le k \le n$. Note that $f(q)$ of degree n is symmetric if and only if

$$q^n f(1/q) = f(q). \tag{8}$$

Theorem 3.5. *We have $D_n([1243]; q) = D_n([1342]; q)$ where, for $n \ge 2$,*

$$D_n([1342]; q) = 2q(1+q)^{n-2} - q \cdot \frac{1 - q^{n-1}}{1 - q}$$

is symmetric.

Proof. It is easy to prove from the explicit form of $D_n([1342])$ that it satisfies equation (8) and so is symmetric. So once this is proved, the equality of the two generating functions follows from Lemma 3.1.

We adopt the notation of Lemma 3.3 and let $\sigma_k = n$ where $2 \le k \le n$. We will consider cases depending on whether ρ or τ is empty. If $\rho = \emptyset$ then by Lemma 3.3 (a) and Lemma 3.4 we have that the generating function for the possible linear τ is $(1+q)^{n-3}$. Also, $\mathrm{cdes}[\sigma] = 2 + \mathrm{des}\,\tau$ by the form of σ, so the contribution of such $[\sigma]$ to $D_n([1342])$ is $q^2(1+q)^{n-3}$. In an analogous way, we see that those $[\sigma]$ with $\tau = \emptyset$ yield $q(1+q)^{n-3}$. Adding these, we have a total of $q(1+q)^{n-2}$ so far.

We now assume that ρ, τ are both nonempty so that $3 \le k \le n-1$. By parts (b) and (c) of Lemma 3.3, either ρ must be an increasing subsequence of consecutive integers or τ must be a decreasing one. Using Lemma 3.4 again, we see that in the first subcase a contribution of $q^2(1+q)^{n-k-1}$ is obtained. And in the second, taking into account the descents in τ, the contribution is $q^{n-k+1}(1+q)^{k-3}$. However, these two subcases overlap when ρ is increasing and τ is decreasing. So we must subtract q^{n-k+1}.

Thus we get a grand total of

$$D_n([1342]) = q(1+q)^{n-2} + \sum_{k=3}^{n-1} [q^2(1+q)^{n-k-1} + q^{n-k+1}(1+q)^{k-3} - q^{n-k+1}].$$

Summing the geometric series and simplifying completes the proof. $\quad\square$

For the avoidance class of the increasing (or decreasing) pattern in $[\mathfrak{S}_4]$, we will need another concept. Given sequences ρ and τ of distinct integers, their *shuffle set* is

$$\rho \shuffle \tau = \{\sigma \ : \ |\sigma| = |\rho| + |\tau| \text{ and both } \rho, \tau \text{ are subsequences of } \sigma\}.$$

For example,

$$12 \shuffle 34 = \{1234, 1324, 1342, 3124, 3142, 3412\}.$$

In the statement of the next result we make the usual convention that $\binom{n}{k} = 0$ if $k > n$.

Theorem 3.6. *We have* $D_n([1234]; q) = q^n D_n([1432]; 1/q)$ *where, for* $n \geq 2$,

$$D_n([1432]; q) = q + (2^{n-1} - n)q^2 + \sum_{j \geq 3} \binom{n}{2j-1} q^j.$$

Proof. We use Callan's description of the avoidance for [1234] translated by complementation to apply to [1432]. We are going to derive a recursion for $D_n([1432]; q)$. If $[\sigma] \in \mathfrak{S}_n[1432]$ then suppose $\sigma_n = 1$ and $\sigma_k = 2$ for some $1 \leq k \leq n-1$. There are three cases.

If $k = 1$ then there is a bijection between such $[\sigma]$ and $\mathrm{Av}_{n-1}[1432]$ obtained by removing 1 and taking the order isomorphic cyclic permutation on $[n-1]$. Since 2 immediately follows 1 cyclically in $[\sigma]$, the descent into 1 remains a descent after applying the map. So the contribution of this case is $D_{n-1}([1432]; q)$.

Now suppose that $2 \leq k \leq n-1$ and write

$$\sigma = \rho 2 \tau 1,$$

where $|\rho| = k - 1$, $|\tau| = n - k - 1$. As Callan proves, ρ must be increasing. So there are two more cases depending upon whether the elements of ρ are consecutive or not. Suppose first that they are not consecutive. In this case, τ must also be increasing so $\mathrm{cdes}[\sigma] = 2$. To compute the number of such σ, note that once the elements of ρ have been picked from $[3, n]$, all of σ is determined. The total number of nonempty subsets of this interval is $2^{n-2} - 1$. And those which consist of consecutive integers are determined by their minimum and maximum element, which could be equal. So there are $\binom{n-1}{2}$ subsets to exclude. The contribution of this case is then

$$\left(2^{n-2} - \binom{n-1}{2} - 1\right) q^2.$$

Finally we consider the case when $\rho \neq \emptyset$ is consecutive (and still increasing), say with minimum $m + 1$ and maximum $M - 1$. Note that if $l = |\tau|$ then $0 \leq l \leq n - 3$. Callan shows that the possible τ are the elements of $(34 \ldots m) \shuffle (M, M+1, \ldots, n)$. Since a permutation can be written as a shuffle in many ways, the same shuffle could occur for different ρ. So it will be convenient to color the elements of the second sequence by marking them with a hat. Thus the σ in this case are in bijection with colored shuffles $(34 \ldots m) \shuffle (\widehat{M}, \widehat{M+1}, \ldots, \widehat{n})$. It will also be convenient to consider these as corresponding to the sequences 2τ by prepending a 2 to each shuffle and considering 2 as an uncolored element. Set S be the set of such sequences $s = 2s_2 s_3 \ldots s_{l+1}$ where l, m, M are allowed to vary over all possible values. Note that if s corresponds to σ then $\mathrm{des}\,\sigma = 2 + \mathrm{des}\,s$. To compute $\mathrm{des}\,s$, we consider the transition indices

$$\mathrm{Tr}\,s = \{i \mid s_i \text{ is colored and } s_{i+1} \text{ is not, or vice-versa}\}.$$

For example, if $s = 23\widehat{6}45\widehat{7}\widehat{8}$ then $\mathrm{Tr}\,s = \{2, 3, 5\}$. It is easy to see that the map $\mathrm{Tr} : S \to 2^{[l]}$, the range being all subsets of $[l]$, is a bijection. Also, every other transition index of s starting with the second corresponds to a descent. So, using the round down function, $\mathrm{des}\,s = \lfloor \# \mathrm{Tr}\,s / 2 \rfloor$. We can now complete this case using $i = \# \mathrm{Tr}\,s$ to see that the contribution is

$$\sum_{l=0}^{n-3} \sum_{i=0}^{l} \binom{l}{i} q^{\lfloor i/2 \rfloor + 2} = \sum_{i=0}^{n-3} q^{\lfloor i/2 \rfloor + 2} \sum_{l=i}^{n-3} \binom{l}{i}$$

$$= \sum_{i=0}^{n-3} \binom{n-2}{i+1} q^{\lfloor i/2 \rfloor + 2}$$

$$= q^2 \sum_{j \geq 0} \left[\binom{n-2}{2j+1} + \binom{n-2}{2j+2} \right] q^j$$

$$= q^2 \sum_{j \geq 0} \binom{n-1}{2j+2} q^j.$$

Putting all the cases together we have

$$D_n([1432]; q) = D_{n-1}([1432]; q) + q^2 \left[2^{n-2} - \binom{n-1}{2} - 1 + \sum_{j \geq 0} \binom{n-1}{2j+2} q^j \right].$$

As usual, the routine verification that our desired formula satisfies this recursion and the initial condition is left to the reader. $\square$

We now turn to the cyclic descent polynomials for pairs in $[\mathfrak{S}_4]$. To simplify notation, for any polynomial $f(q)$ and $n \in \mathbb{N}$ we let

$$f^{(n)}(q) = q^n f(1/q).$$

Theorem 3.7. *We have the following descent polynomials.*

(a) *We have*

$$D_n([1234],[1243]) = D_n([1342],[1432]) = D_n^{(n)}([1243],[1432]) = D_n^{(n)}([1234],[1342]).$$

And for $n \geq 3$

$$D_n([1234],[1342];q) = (2n-5)q^{n-2} + q^{n-1}.$$

(b) *We have*

$$D_n([1423],[1432]) = D_n^{(n)}([1234],[1324]).$$

And for $n \geq 3$

$$D_n([1234],[1324];q) = (2n-5)q^{n-2} + q^{n-1}.$$

(c) *We have*

$$D_n([1324],[1432]) = D_n^{(n)}([1234],[1423]).$$

And for $n \geq 1$

$$D_n([1234],[1423];q) = q^{n-1} + \binom{n-1}{2}q^{n-2}.$$

(d) *We have*

$$D_n([1243],[1423]) = D_n([1342],[1423]) = D_n^{(n)}([1243],[1324]) = D_n^{(n)}([1324],[1342]).$$

And for $n \geq 1$

$$D_n([1324],[1342];q) = q + \sum_{k=2}^{n-1}(n-k)q^k.$$

(e) *For $n \geq 4$ we have*

$$D_n([1243],[1342];q) = q + q^2 + q^{n-1} + q^{n-2}.$$

(f) *For $n \geq 3$ we have*

$$D_n([1324], [1423]; q) = q(1+q)^{n-2}.$$

Proof. We will only prove (a) as the others follow easily in a similar fashion from the descriptions of the avoidance classes in Section 2. We adopt the notation of the proof of Theorem 2.2.

We will use the description of the generating tree to obtain a recursion for $D_{n+1}([1243], [1432])$. Note that if $n + 1$ is inserted in site i of σ to form σ' then

$$\text{cdes}[\sigma'] = \begin{cases} \text{cdes}[\sigma] & \text{if } i \text{ is a cyclic descent,} \\ \text{cdes}[\sigma] + 1 & \text{if } i \text{ is a cyclic ascent.} \end{cases}$$

Since the site before n is always active, these children will give a contribution of $qD_n([1243], [1432])$ because such a site is a cyclic ascent. In δ and ϵ, insertion in the other active site gives permutations with $n - 1$ descents. So

$$D_{n+1}([1243], [1432]) = 2q^{n-1} + qD_n([1243], [1432]).$$

It is now easy to check that the formula in (a) satisfies this recursion and is also valid at $n = 3$, completing the proof. $\square$

For classes avoiding 3 or more patterns, we will only write down the results for those which are not eventually constant. The interested reader can easily compute the polynomials for the remaining classes. We also content ourselves with stating the polynomial for one member of every trivial Wilf equivalence class since the rest can be computed from Lemma 3.1.

Theorem 3.8. *We have the descent polynomials*

$$D_n([1234], [1342], [1423]; q) = D_n([1234], [1324], [1423]; q) = (n-2)q^{n-2} + q^{n-1}$$

and

$$D_n([1324], [1342], [1423]; q) = q \cdot \frac{1 - q^{n-1}}{1 - q}$$

for $n \geq 2$. $\square$

4. Consecutive patterns

We will now concentrate on the consecutive case. For the rest of this section, we let $\pi = \underline{\pi_1 \pi_2 \ldots \pi_k}$ be a consecutive pattern. We will relate the number of occurrences of π in linear permutations to the number of occurrences of $[\pi]$ in cyclic permutations.

We let $o_\pi(\sigma)$ be the number of occurrences of π in a linear permutation σ. Similarly, we denote by $c_\pi[\sigma]$ the number of occurrences of $[\pi]$ in $[\sigma]$. This number is well defined, in the sense that it does not depend on the chosen representative of $[\sigma]$, since rotating σ simply changes the positions of the occurrences of $[\pi]$, but not the actual subsequences or how many there are. Note also that $c_\pi[\sigma] = 0$ precisely if $[\sigma]$ avoids $[\pi]$. For example, $c_{132}[25314] = 2$, since [253] and [142] are occurrences of $[\sigma]$ in $[\pi]$. On the other hand, $c_{132}[24531] = 0$, so $[24531] \in \mathrm{Av}_5[\underline{132}]$.

We denote by

$$P_\pi(u, z) = \sum_{n \geq 0} \sum_{\sigma \in \mathfrak{S}_n} u^{o_\pi(\sigma)} \frac{z^n}{n!}$$

the exponential generating function counting occurrences of a consecutive pattern π in linear permutations, and let $\omega_\pi(u, z) = 1/P_\pi(u, z)$. Formulas and differential equations for $P_\pi(u, z)$ and $\omega_\pi(u, z)$, for various patterns π, have been given in [22,23], see also [29, 34,32,18] for related work.

Let

$$C_\pi(u, z) = \sum_{n \geq 0} \sum_{[\sigma] \in [\mathfrak{S}_n]} u^{c_\pi[\sigma]} \frac{z^n}{n!} \tag{9}$$

be the exponential generating function counting occurrences of $[\pi]$ in cyclic permutations, and note that

$$C_\pi(0, z) = \sum_{n \geq 0} \#\mathrm{Av}_n[\pi] \frac{z^n}{n!}.$$

As in the case of consecutive patterns in linear permutations, letting

$$\pi^r = \underline{\pi_k \dots \pi_2 \pi_1}$$

and

$$\pi^c = \underline{(k + 1 - \pi_1)(k + 1 - \pi_2) \dots (k + 1 - \pi_k)},$$

it is clear that

$$C_\pi(u, z) = C_{\pi^c}(u, z) = C_{\pi^r}(u, z) = C_{\pi^{rc}}(u, z),$$

since occurrences of $[\pi]$ in $[\sigma]$ correspond to occurrences of $[\pi^r]$ in $[\sigma^r]$, and to occurrences of $[\pi^c]$ in $[\sigma^c]$. For example, for patterns of length 3, we have $C_{123}(u, z) = C_{321}(u, z)$ and $C_{132}(u, z) = C_{312}(u, z) = C_{213}(u, z) = C_{231}(u, z)$.

For a function $F(u, z)$, we will use $F'(u, z)$ to denote its partial derivative with respect to the variable z. Our central result in this section relates consecutive patterns in the

cyclic case with those in the linear case. The requirement $\pi_1 = 1$ can be replaced, by the above symmetries, with any of $\pi_1 = k$, $\pi_k = 1$, or $\pi_k = k$, where k is the length of π.

For the purposes of the proof we will let a *permutation* be any linear or cyclic ordering of a finite set of positive integers. Any set of cyclic permutations $[\Sigma] = \{[\sigma^{(1)}], [\sigma^{(2)}], \ldots, [\sigma^{(m)}]\}$ will be given *weight*

$$\mathrm{wt}[\Sigma] = u^{c_\pi[\sigma^{(1)}]} \cdot u^{c_\pi[\sigma^{(2)}]} \cdots u^{c_\pi[\sigma^{(m)}]},$$

and any linear permutation σ will be given weight $\mathrm{wt}\,\sigma = u^{o_\pi(\sigma)}$. Finally, the *left-right minima* of $\sigma = \sigma_1\sigma_2\ldots\sigma_n$ are the elements σ_i such that

$$\sigma_i = \min\{\sigma_1, \sigma_2, \ldots, \sigma_i\}.$$

These elements give rise to the *left-right minima factorization* of σ, which is

$$\sigma = \sigma^{(1)}\sigma^{(2)} \ldots \sigma^{(m)} \tag{10}$$

where $\sigma^{(i)}$ is the factor (consecutive subword) of σ starting at the ith left-right minimum and ending just before the $(i+1)$st.

Theorem 4.1. *Let* $\pi = \underline{\pi_1\pi_2\ldots\pi_k}$ *be a consecutive pattern with* $\pi_1 = 1$. *Then*

$$C_\pi(u, z) = 1 + \ln P_\pi(u, z).$$

Proof. Exponentiating the equation in the statement of the theorem, it suffices to prove that

$$P_\pi(u, z) = e^{C_\pi(u,z)-1}.$$

By the Exponential Formula (see Theorem 4.5.1 in Sagan's book [36]), it suffices to show that there is a bijection ϕ between permutations $\sigma \in \mathfrak{S}_n$ and sets of cyclic permutations

$$[\Sigma] = \{[\sigma^{(1)}], [\sigma^{(2)}], \ldots, [\sigma^{(m)}]\}$$

such that

(a) $\biguplus_{i=1}^{k} \sigma^{(i)} = \{1, 2, \ldots, n\}$, the union being of the underlying sets of the $\sigma^{(i)}$, and
(b) $\mathrm{wt}\,\sigma = \mathrm{wt}[\Sigma]$.

Define

$$\phi(\sigma) = \{[\sigma^{(1)}], [\sigma^{(2)}], \ldots, [\sigma^{(m)}]\}$$

where the $\sigma^{(i)}$ are the factors in (10). Then (a) holds because every element of $\{1, 2, \ldots, n\}$ must appear in exactly one of the factors of the factorization. To prove (b), let us show that any occurrence of π in σ is entirely contained in one of the $\sigma^{(i)}$. Indeed, if the occurrence overlaps two or more factors, then the left-right minimum of the second factor is smaller than the first element of the occurrence. This contradicts the fact that π begins with 1.

To show ϕ is bijective, we construct its inverse. Given $[\Sigma]$, rotate each cyclic permutation so that $\sigma^{(i)}$ starts with its minimum element. Then concatenate these linear permutations in order of decreasing first element to form σ. It is easy to check that this describes the inverse of ϕ.   □

Expressions for $P_\pi = P_\pi(u, z)$ are known for certain consecutive patterns π, often in the form of differential equations satisfied by its reciprocal $\omega_\pi = 1/P_\pi$. In fact, up to symmetry, all the patterns σ for which explicit differential equations have been found so far satisfy $\sigma_1 = 1$. Thus, Theorem 4.1 can be applied to these patterns to deduce an expression for $C_\pi = C_\pi(u, z)$.

Restating Theorem 4.1 to relate C_π and ω_π, we have $C_\pi = 1 - \ln \omega_\pi$, from where $C_\pi' = -\omega_\pi'/\omega_\pi$, and

$$\omega_\pi = e^{1 - C_\pi}. \tag{11}$$

In some cases, this relation allows us to obtain differential equations directly in terms of C_π, as we will see below.

It is proved in [22, Theorem 3.1] (see also [23, Theorem 2.1]) that, for $\pi = \underline{12 \ldots k}$ with $k \geq 3$, the function $\omega_\pi = \omega_\pi(u, z)$ satisfies the differential equation

$$\omega_\pi^{(k-1)} + (1 - u)(\omega_\pi^{(k-2)} + \cdots + \omega_\pi' + \omega_\pi) = 0 \tag{12}$$

with initial conditions $\omega_\pi(u, 0) = 1$, $\omega_\pi'(u, 0) = -1$, and $\omega_\pi^{(i)}(u, 0) = 0$ for $2 \leq i \leq k - 2$. In [23, Theorem 2.4], similar differential equations are given for ω_π whenever π is a so-called *chain pattern* (see [23, Definition 2.2]). Chain patterns generalize monotone patterns, but they still satisfy $\pi_1 = 1$ (up to symmetry), as shown in [23, Lemma 2.3]. Thus, for all such patterns π, Theorem 4.1 can be used to determine $C_\pi = 1 - \ln \omega_\pi$.

It is possible to rewrite (12) as a differential equation for C_π using the identity (11). For example, when $k = 3$, we obtain the following.

Corollary 4.2. *Let $R = R_{\underline{123}}(u, z) = C'_{\underline{123}}(u, z)$. Then R satisfies the differential equation*

$$R' = R^2 + (u - 1)(R - 1) \tag{13}$$

with initial condition $R(u, 0) = 1$. An explicit expression is given by

$R_{\underline{123}}(u, z)$

$$= \frac{1}{2} \left(1 - u - \tanh\left(\frac{z\sqrt{u^2 + 2u - 3}}{2} - \operatorname{arctanh}\left(\frac{u+1}{\sqrt{u^2 + 2u - 3}} \right) \right) \sqrt{u^2 + 2u - 3} \right),$$

which, for $u = 0$, simplifies to

$$R_{\underline{123}}(0, z) = \frac{1}{2} + \frac{\sqrt{3}}{2} \tan\left(\frac{\sqrt{3}}{2} z + \frac{\pi}{6} \right).$$

Proof. Differentiating Equation (11), we get $\omega'_\pi = -C'_\pi e^{1-C_\pi}$ and $\omega''_\pi = \left(-C''_\pi + (C'_\pi)^2 \right) e^{1-C_\pi}$. Substituting these expressions into Equation (12) for $k = 3$, and dividing both sides by e^{1-C_π}, we obtain Equation (13). □

Setting $u = 0$ in Equation (13) gives

$$R'_{\underline{123}}(0, z) = R_{\underline{123}}(0, z)^2 - R_{\underline{123}}(0, z) + 1, \tag{14}$$

proving a conjecture from an earlier version of this paper. For $k = 4$, a similar computation yields the following.

Corollary 4.3. *Let $R = R_{\underline{1234}}(u, z) = C'_{\underline{1234}}(u, z)$. Then R satisfies the differential equation*

$$R'' = 3R'R - R^3 + (u - 1)(R' - R^2 + R - 1) \tag{15}$$

with initial conditions $R(u, 0) = 1$, $R'(u, 0) = 1$. For $u = 0$, an explicit expression is given by

$$R_{\underline{1234}}(0, z) = \frac{\cos z + \sin z + e^{-z}}{\cos z - \sin z + e^{-z}}.$$

In the case of linear permutations, explicit expressions for $P_{\underline{123}}(u, z)$, $P_{\underline{123}}(0, z)$ and $P_{\underline{1234}}(0, z)$ have been given in [22, Theorems 4.1 and 4.3]. Let us also point out that, for $\sigma = \underline{12\ldots k}$, the generating function $R_\pi = C'_\pi$ coincides with the generating function denoted by R in the proof of [22, Theorem 3.1].

A consecutive pattern π of length k is called *non-overlapping* if two occurrences of σ cannot overlap in more than one position; in other words, there is no permutation $\sigma \in \mathfrak{S}_{2k-2}$ with $o_\pi(\sigma) \geq 2$.

Generalizing [22, Theorem 3.2], it is shown in [23, Theorem 3.1] that, for any non-overlapping consecutive pattern π of length $k \geq 3$ with $\pi_1 = 1$, the function $\omega_\pi = \omega_\pi(u, z)$ satisfies the following differential equation, where $b = \sigma_k$:

$$\omega_\pi^{(b)} + (1 - u) \frac{z^{k-b}}{(k-b)!} \omega'_\pi = 0, \tag{16}$$

with initial conditions $\omega_\pi(u,0) = 1$, $\omega'_\pi(u,0) = -1$, and $\omega_\pi^{(i)}(u,0) = 0$ for $2 \leq i \leq b-1$. Again, by Theorem 4.1, this determines $C_\pi = 1 - \ln \omega_\pi$ for all such patterns. In this case, the generating function C'_π coincides with the generating function denoted by R in the proof of [22, Theorem 3.2].

In the case $b = 2$, rewriting (16) as a differential equation for C_π using (11) and its derivatives, we obtain the following.

Corollary 4.4. *Let π be a non-overlapping pattern of length $k \geq 3$ with $\pi_1 = 1$ and $\pi_k = 2$, and let $R = R_\pi(u,z) = C'_\pi(u,z)$. Then R satisfies the differential equation*

$$R' = R^2 + (u-1)\frac{z^{k-2}}{(k-2)!}\, R \tag{17}$$

with initial condition $R(u,0) = 1$. An explicit expression is given by

$$R_\sigma(u,z) = \frac{e^{(u-1)\frac{z^{k-1}}{(k-1)!}}}{1 - \int_0^z e^{(u-1)\frac{t^{k-1}}{(k-1)!}}\, dt},$$

or equivalently,

$$C_\pi(u,z) = 1 - \ln\left(1 - \int_0^z e^{(u-1)\frac{t^{k-1}}{(k-1)!}}\, dt\right).$$

Setting $u = 0$ in Equation (17) for $k = 3$ gives the equation

$$R'_{\underline{132}}(0,z) = R_{\underline{132}}(0,z)^2 - zR_{\underline{132}}(0,z).$$

Dividing both sides by $R_{\underline{132}}(0,z)$, integrating, and using that $R_{\underline{132}} = C'_{\underline{132}}$, we obtain $\ln C'_{\underline{132}}(0,z) = C_{\underline{132}}(0,z) - z^2/2$, or equivalently,

$$C'_{\underline{132}}(0,z) = e^{C_{\underline{132}}(0,z) - z^2/2}, \tag{18}$$

proving a conjecture in an earlier version of this paper.

In [23], differential equations are also given for $\omega_\pi(u,z)$ when π is any of $\underline{1324}$, $\underline{12534}$, or $\underline{13254}$. For each of these patterns, Theorem 4.1 can again be applied to obtain $C_\pi(u,z)$.

5. Open problems and concluding remarks

We collect here various areas for future research in the hopes that the reader will be interested in pursuing this work.

5.1. Longer patterns

There has been very little work about containment and avoidance for cyclic patterns of length longer than 4. Of course, the cyclic Erdős–Szekeres Theorem, Theorem 1.3 above, is one such result. There is also a paper of Gray, Lanning and Wang [27] where the authors consider cyclic packing (maximizing the number of copies of a given pattern among all the permutations $[\sigma] \in [\mathfrak{S}_n]$ for some n) and superpatterns (permutations containing all the patterns $[\pi] \in [\mathfrak{S}_k]$ for some k). It would be interesting to see if there are nice enumerative formulas for classes consisting of cyclic patterns of length 5 and up.

5.2. Other statistics

One could study other cyclic statistics. For example, the *peak set* of a linear permutation is

$$\mathrm{Pk}\,\pi = \{i \mid \pi_{i-1} < \pi_i > \pi_{i+1}\}$$

with corresponding *peak number*

$$\mathrm{pk}\,\pi = \#\,\mathrm{Pk}\,\pi.$$

Peaks are an important part of Stembridge's theory of enriched P-partitions [40] where P is a partially ordered set. On the enumerative side, the study of permutations which have a given peak set has been a subject of current interest [5–7,11,14–16]. Now define the *cyclic peak number* to be

$$\mathrm{cpk}[\pi] = \#\{i \mid \pi_{i-1} < \pi_i > \pi_{i+1} \text{ where subscripts are taken modulo } n\}.$$

As with cdes, this is well defined since it is independent of the choice of representative of $[\pi]$. There should be interesting generating functions for the distribution of cpk over avoidance classes, or even for the joint distribution of cdes and cpk. As evidence, we prove one such result.

Theorem 5.1. *For $n \geq 3$*

$$\sum_{[\sigma] \in \mathrm{Av}_n([1234],[1342])} q^{\mathrm{cdes}[\sigma]} t^{\mathrm{cpk}[\sigma]} = q^{n-2}t + (2n-6)q^{n-2}t^2 + q^{n-1}t.$$

Proof. Let $F_n(q,t)$ denote the desired generating function. We proceed as in the proof of Theorem 3.7 (a) to find a recursion for $F_{n+1}(q,t)$. Since the largest element of $[\sigma]$ is always a cyclic peak, inserting $n+1$ before n does not change cpk. So this contributes $qF_n(q,t)$ to the recursion. For δ and ϵ, inserting $n+1$ in the other active site

increases the number of peaks to 2. So the contribution from these cases is $2q^{n-1}t^2$. In summary

$$F_{n+1}(q,t) = 2q^{n-1}t^2 + qF_n(q,t)$$

and the desired polynomial is easily seen to be the solution. $\quad\square$

In a recent paper Adin, Gessel, Reiner, and Roichman [1] defined a cyclic analogue of the Hopf algebra of quasisymmetric functions. In this context the cyclic descent set of a linear permutation arises naturally in the description of the product in this algebra. They also raise the following intriguing question.

Question 5.2. *Find an analogue of the major index for cyclic permutations that has nice properties, such as a generating function over $[\mathfrak{S}_n]$ which factors nicely as does the generating function for the ordinary major index over $\mathfrak{S}_n$.*

5.3. Vincular patterns

We will show how one vincular class is enumerated by the Catalan numbers. As remarked in the introduction [31], Li has continued our work with an extensive study of vincular pattern avoidance.

Theorem 5.3. *We have*

$$[13\underline{24}] \equiv [1\underline{42}3] \equiv [\underline{13}24] \equiv [2\underline{31}4].$$

And for $n \geq 1$

$$\# \mathrm{Av}_n[13\underline{24}] = C_{n-1}.$$

Proof. The Wilf equivalences are trivial. To prove the Catalan formula, suppose that $[\sigma] \in \mathrm{Av}_n[13\underline{24}]$ for $n \geq 2$ and write σ so that $\sigma_n = n$ and $\sigma_{n-1} = m$ for some $m \in [n-1]$. First notice that $\sigma = \rho\tau mn$ where ρ and τ are permutations of $[m+1, n-1]$ and $[m-1]$, respectively. For if there are $x < m < y < n$ with x before y in σ then $[xymn]$ is a copy of $[13\underline{24}]$. Furthermore, it is clear that $[m\rho]$ and $[\tau m]$ must avoid the forbidden pattern.

We claim the if $\sigma = \rho\tau mn$ where ρ and τ obey the restrictions of the previous paragraph then $[\sigma]$ avoids $[13\underline{24}]$. Suppose, towards a contradiction, that a copy $[\kappa] = [wyxz]$ exists with $wyxz$ order isomorphic to $13\underline{24}$. Consider the elements x and z which play the roles of 2 and 4. The possibility that they are m and n, respectively, is ruled out by the fact that every element of ρ is larger than every element of τ. If $z \in \tau m$ then all of κ must be in this subsequence since z is the largest element of the copy. But this is impossible since $[\tau m]$ avoids the bad pattern. Finally, suppose $z \in \rho$. This

forces $x \in \rho$ since it comes cyclically just before z, and n is too large to be x. We must also have $y \in \rho$ since $x < y < z$. But now there is no possible choice for w. Indeed, if $w \in [m\rho]$ then $[\kappa]$ is in this subsequence, contradicting our assumption. And if $w \in \tau$ then it could be replaced by m since $x, y, z > m$, yielding the same contradiction as before.

From the first two paragraphs we immediately get the recursion

$$\# \operatorname{Av}_n[13\underline{24}] = \sum_{m=1}^{n-1} \# \operatorname{Av}_m[13\underline{24}] \cdot \# \operatorname{Av}_{n-m}[13\underline{24}].$$

From this the Catalan enumeration follows by induction. $\square$

For the case of consecutive patters, a natural problem for further research would be to find $C_\pi(u, z)$ for consecutive patterns π that do not begin with 1 (even after applying the basic symmetries).

In a different direction, it is shown in [19] that, for n large enough, the number of (linear) permutations in $\mathfrak{S}_n$ that avoid a consecutive pattern π of length k is largest when π is a monotone pattern, and it is smallest when $\pi = \underline{12\ldots(k-2)k(k-1)}$ (or any of its symmetries). One could ask if there is an analogue of this theorem for consecutive patterns in cyclic permutations.

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